Chapter 5: Probability: What are the Chances?

Section 5.2
Probability Rules

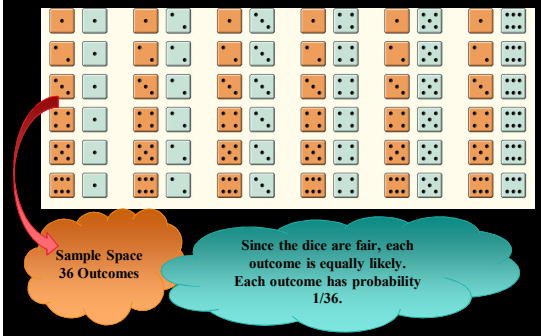
Probability Models
In Section 5.1, we used simulation to imitate chance behavior. Fortunately, we don't have to always rely on simulations to determine the probability of a particular outcome.

Descriptions of chance behavior contain two parts:

Definition:
The **sample space *S*** of a chance process is the set of all possible outcomes.

A **probability model** is a description of some chance process that consists of two parts: a sample space *S* and a probability for each outcome.

Example 1: Many board games involve rolling dice. Imagine rolling two fair, six-sided dice—one that's red and one that's green. Give a probability model for this chance process.



Probability Models
Probability models allow us to find the probability of any collection of outcomes.

Definition:
An **event** is any collection of outcomes from some chance process. That is, an event is a subset of the sample space. Events are usually designated by capital letters, like *A, B, C,* and so on.

If *A* is any event, we write its probability as *P*(*A*).

In the dice-rolling example, suppose we define event *A* as "sum is 5."

What's *P*(*A*), the probability that event *A* occurs?
There are 4 outcomes that result in a sum of 5.
Since each outcome has probability 1/36, *P*(*A*) = 4/36.

Suppose event *B* is defined as "sum is not 5." What is *P*(*B*)? *P*(*B*) = 1 – 4/36 = 32/36

Example 2: Imagine flipping a fair coin three times. Give a probability model for this chance process.

Basic Rules of Probability
All probability models must obey the following rules:

**The probability of any event is a number between 0 and 1.* The probability of an event is the long-run proportion of repetitions on which that event occurs. Any proportion is a number between 0 and 1, so any probability is also a number between 0 and 1. An event with probability 0 never occurs, and an event with probability 1 occurs on every trial. An event with probability 0.5 occurs in half the trials in the long run.

*All possible outcomes together must have probabilities whose sum is 1. Because some outcome must occur on every trial, the sum of the probabilities for all possible outcomes must be exactly 1.

*If all outcomes in the sample space are equally likely, the probability that event A occurs can be found using the formula

$$P(A) = \frac{\text{number of outcomes corresponding to event } A}{\text{total number of outcomes in sample space}}$$

Basic Rules of Probability

All probability models must obey the following rules:

*The probability that an event does not occur is 1 minus the probability that the event does occur. If an event occurs in (say) 70% of all trials, it fails to occur in the other 30%. The probability that an event occurs and the probability that it does not occur always add to 100%, or 1. We refer to the event "not A" as the complement of A and denote it by AC.

*If two events have no outcomes in common, the probability that one or the other occurs is the sum of their individual probabilities. If one event occurs in 40% of all trials, a different event occurs in 25% of all trials, and the two can never occur together, then one or the other occurs on 65% of all trials because 40% + 25% = 65%.

Definition:
Two events are **mutually exclusive (disjoint)** if they have no outcomes in common and so can never occur together.

We can summarize the basic probability rules more concisely in symbolic form.

- For any event A, $0 \le P(A) \le 1$.
- If S is the sample space in a probability model, then $P(S) = 1$.
- In the case of equally likely outcomes,

$$P(A) = \frac{\text{number of outcomes corresponding to event } A}{\text{total number of outcomes in sample space}}$$

- **Complement rule:** $P(A^C) = 1 - P(A)$
- **Addition rule for mutually exclusive events:** If A and B are mutually exclusive,

$$P(A \text{ or } B) = P(A) + P(B).$$

Example 3: Randomly select a student who took the 2013 AP® Statistics exam and record the student's score. Here is the probability model:

Score:	1	2	3	4	5
Probability:	0.235	0.188	0.249	0.202	0.126

a) Show that this is a legitimate probability model.

Each probability is between 0 and 1 and
0.235 + 0.188 + 0.249 + 0.202 + 0.126 = 1

b) Find the probability that the chosen student scored 3 or better.

There are two ways to find this probability:

By the addition rule, *P*(3 or better) = 0.249 + 0.202 + 0.126 = 0.577.

By the complement rule and addition rule,
***P*(3 or better) = 1 – *P*(2 or less) = 1 – (0.235 + 0.188) = 1 – 0.423 = 0.577.**

Two-Way Tables and Probability

When finding probabilities involving two events, a two-way table can display the sample space in a way that makes probability calculations easier.

Example 4: Students in a college statistics class wanted to find out how common it is for young adults to have their ears pierced. They recorded data on two variables—gender and whether the student had a pierced ear—for all 178 people in the class. The two-way table below displays the data.

	Pierced Ears?		
Gender	Yes	No	Total
Male	19	71	90
Female	84	4	88
Total	103	75	178

Suppose we choose a student from the class at random. Find the probability that the student

a) has pierced ears.

Define events *A*: is male and *B*: has pierced ears.

Each student is equally likely to be chosen. 103 students have pierced ears.
So, *P*(pierced ears) = *P*(*B*) = 103/178.

b) is a male with pierced ears.

	Pierced Ears?		
Gender	Yes	No	Total
Male	19	71	90
Female	84	4	88
Total	103	75	178

Define events *A*: is male and *B*: has pierced ears.

We want to find *P*(male *and* pierced ears), that is, *P*(*A* and *B*). Look at the intersection of the "Male" row and "Yes" column. There are 19 males with pierced ears. So, *P*(*A* and *B*) = 19/178.

c) is a male or has pierced ears.

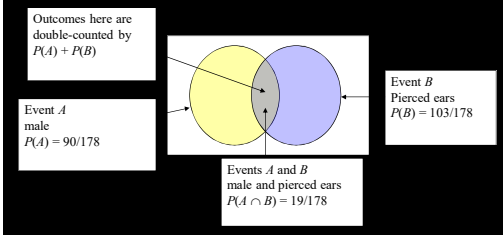
	Pierced Ears?		
Gender	Yes	No	Total
Male	19	71	90
Female	84	4	88
Total	103	75	178

Define events *A*: is male and *B*: has pierced ears.

We want to find *P*(male *or* pierced ears), that is, *P*(*A* or *B*). There are 90 males in the class and 103 individuals with pierced ears. However, 19 males have pierced ears – don't count them twice!
P(*A* or *B*) = (19 + 71 + 84)/178. So, *P*(*A* or B*)* = 174/178

Note, the previous example illustrates the fact that we can't use the addition rule for mutually exclusive events unless the events have no outcomes in common.

The **Venn diagram** below illustrates why.



General Addition Rule for Two Events

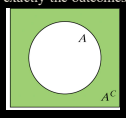
If A and B are any two events resulting from some chance process, then

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

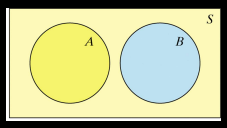
Venn Diagrams and Probability

Because Venn diagrams have uses in other branches of mathematics, some standard vocabulary and notation have been developed.

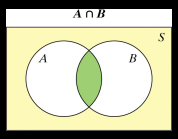
The complement A^C contains exactly the outcomes that are not in A.



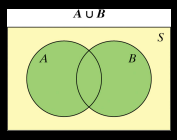
The events A and B are mutually exclusive (disjoint) because they do not overlap. That is, they have no outcomes in common.



The intersection of events A and B ($A \cap B$) is the set of all outcomes in both events A and B.

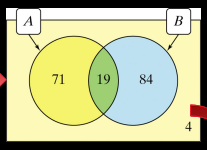


The union of events A and B ($A \cup B$) is the set of all outcomes in either event A or B.



Example 5: In the preceding example, we looked at data from a survey on gender and ear piercings for a large group of college students. The chance process came from selecting a student in the class at random. Our events of interest were A: is male and B: has pierced ears. Here is the two-way table that summarizes the sample space:

	Pierced Ears?		
Gender	Yes	No	Total
Male	19	71	90
Female	84	4	88
Total	103	75	178



Define events A: is male and B: has pierced ears.

Region in Venn diagram	In words	In symbols	Count
In the intersection of two circles	Male and pierced ears	$A \cap B$	19
Inside circle A, outside circle B	Male and no pierced ears	$A \cap B^C$	71
Inside circle B, outside circle A	Female and pierced ears	$A^C \cap B$	84
Outside both circles	Female and no pierced ears	$A^C \cap B^C$	4

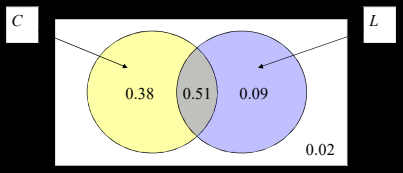
Example 6: According to the National Center for Health Statistics, in December 2012, 60% of U.S. households had a traditional landline telephone, 89% of households had cell phones, and 51% had both. Suppose we randomly selected a household in December 2012.

a) Make a two-way table that displays the sample space of this chance process.

We will define events L: has a landline and C: has a cell phone.

	Cell phone	No cell phone	Total
Landline	0.51	0.09	0.60
No landline	0.38	0.02	0.40
Total	0.89	0.11	1.00

b) Construct a Venn diagram to represent the outcomes of this chance process.



c) Find the probability that the household has at least one of the two types of phones.

To find the probability that the household has at least one of the two types of phones, we need to find the probability that the household has a landline, a cell phone, or both.

$$P(L \cup C) = P(L) + P(C) - P(L \cap C) = 0.60 + 0.89 - 0.51 = 0.98$$

There is a 0.98 probability that the household has at least one of the two types of phones.

d) Find the probability that the household has a cell phone only.

P(cell phone only) = $P(L^C \cap C) = 0.38$. There is a 0.38 probability that the household is cell-phone only.

AP EXAM TIP: Many probability problems involve simple computations that you can do on your calculator. It may be tempting to just write down your final answer without showing the supporting work. Don't do it! A "naked answer," even if it's correct, will usually earn you no credit on a free-response question.